**Heisenberg Formulation of Quantum Mechanics**

In the last file we made parenthetical comments about the commutation relations satisfied by [x,p] and [Li, Lj]. We can in fact takes these commutation relations as a postulate, and from that derive the representation of these operators…

**Derivation of the representation of p from commutation relation**

So start with:



And now we actually introduce another operator, called the translation operator, defined via:



And so we can see that the translation operator simply maps one base position base ket to another a distance Δx away. For infinitesimal displacements we can make a Taylor series expansion of the translation operator and say that:



So that for infinitesimal displacements we have:



Now observe that  (the identity operator), and let’s call to be some operator and so write:



Now let’s look at the commutation relation between  and . We have:



So we see that:



which is very close to the commutation relation between x and px. Multiplying both sides by *iћ* we have:



and so it must be the case that:



Finally, since we know the action of on |x>, we know implicitly the action of  on |x>. For instance,



and so now we can determine the action of ****on a wavefunction in the coordinate basis:



Multiplying both sides by |x> and integrating we have:



and so it follows that the coordinate representation of the operator should be:



just as we guessed using the Schrodinger picture. We can repeat this argument for the y and z directions, and so it follows that:



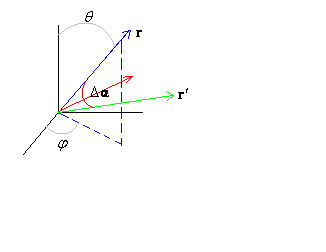
And now that we have this, the representation of all other operators can be constructed.

**Representation of angular momentum operator, L**

In particular, we can get **L** = **r**×**p**, but it is instructional to get this without recourse to **p** and instead make a similar analysis to that which we employed in determining the representation of **p**. To that end, let’s introduce the rotation operator D(Δ**α**) defined by:



**r**′ is the vector obtained by rotating **r** by Δ**α**.



For instance, let’s consider the rotation operator in the z-direction.



To figure out the new θ′ and φ′ we can use a little bit of linear algebra. The rotation matrices in 3D space are given as follows:



And the position vector |**r**> corresponds to the column vector:



So to figure out what a rotation would do to |**r**> we can apply the rotation matrix to **r**. Let’s look at what a rotation about the z-direction does.



and so we have:



This actually should be fairly obvious from the diagram above. Next lets consider a rotation about the x-axis.



and to get the new angles…



Now to extract the new angles refer to the graph above. The angle θ′ can be determined from the projection of **r**′ on **z**. And the angle φ′ can be determined from the projection of the (normalized) x-y plane component of **r**′ with **x**. So we have:



and so we have:



And finally let’s look at the effect of a rotation about the y-axis.



Now to extract the new angles refer to the graph above. The angle θ′ can be determined from the projection of **r**′ on **z**. And the angle φ′ can be determined from the projection of the (normalized) x-y plane component of **r**′ with **x**. So we have:



and so then we have:



Now like before, we’d like to expand each of these operators in a Taylor series in Δα about 0. So let’s do that with each in turn,



So we can define,



As for the x – direction we have to make a small δα approximation on the angles.



and as for the other,



So we have:



and so we can say:



Finally for the y-operator…



and so we have:



so we can say,



Just like the momentum operator was proportional to the first term in the Taylor expansion of the translation operator, we might suspect that the orbital angular momentum operators are the terms proportional to the first terms in the Taylor expansions of the rotation operators. And we would be correct! For example, let’s calculate the commutation relation between Qx and Qy.



after a lot of cancellations. So we have:



which is almost the angular momentum commutation relations. If we multiply all the operators by -*iћ*, then we get them as we’ll have:



which suggests the identification:



which is in fact correct! This can be verified by checking the other 2 commutation relations. Finally, when we calculate the total angular momentum (squared) we get:



It is important to note that the representation of the operators was determined essentially directly from the commutation relations that the angular momentum operators satisfied. Given the commutation relations satisfied, we can ascertain that **L** is the generator of rotation. And from this fact we know how **L** acts on a ket. And from this we can determine the coordinate representation of the operator. Of course if we were still unsure of our prescription, then we could verify that these formulas are identical to the coordinate representation of = × , (which we worked out in Cartesian coordinates in the corresponding Schrodinger file) expressed in spherical coordinates.